(8) and introducing the voltage ratio from Eq. (11) gives

$$\mathcal{P} = 4\mu\bar{u}^2M^2\left(\frac{L}{w}\right)\frac{(K)(1-K)}{2+\theta_1+\theta_2} \tag{12}$$

Equation (12) shows the dependence of the power output on the wall conductances for fixed voltage ratio and mean velocity. For a given K and $\bar{u}$, the power is a maximum when $\theta_1 + \theta_2 = 0$, corresponding to perfectly insulating boundaries.

It is of interest to express the power output in terms of the pressure gradient. A relationship between mean velocity and pressure gradient can be obtained in the form

$$\bar{u} = \frac{(w^2/\mu)(dP/dx)}{[2M^2K/(2+\theta_1+\theta_2)] - [M^3/(M - \tanh M)]} \tag{13}$$

Combining Eqs. (12) and (13) gives

$$\mathcal{P} = \frac{4M^2(L/w)(K)(1-K)(w^4/\mu)(dP/dx)^2}{(2+\theta_1+\theta_2)\left(\dfrac{2M^2K}{2+\theta_1+\theta_2} - \dfrac{M^3}{M-\tanh M}\right)^2} \tag{14}$$

The voltage ratio that maximizes the power output can be obtained by setting the derivative of $\mathcal{P}$ with respect to K equal to zero. It is clear that the optimization can be done at either constant mean velocity or constant pressure gradient. If one of these parameters is held constant while the external loading conditions are changed, the other parameter must adjust itself in a manner compatible with the equation of motion. From Eq. (12), the condition

$$\partial\mathcal{P}/\partial K)_{\bar{u}} = 0$$

gives $K = \frac{1}{2}$ as the optimum voltage ratio at constant mean velocity. From Eq. (14), the condition

$$\partial\mathcal{P}/\partial K)_{dP/dx} = 0$$

gives

$$K = \frac{\frac{1}{2}}{1 + \{(\tanh M - M)/[M(2+\theta_1+\theta_2)]\}}$$

as the optimum voltage ratio at constant pressure gradient. There is thus a distinction between power optimization at constant mean velocity (or mass flow rate) and constant pressure gradient in that the latter case depends on the Hartmann number M and the wall conductances.

A final relationship of usefulness is that between the external load resistance R and the voltage ratio K. From Eq. (7), the total current per unit length of generator can be written as

$$I = 2\sigma\bar{u}B_y w(1-K) \tag{15}$$

The total current also is related to the external load through the relation

$$I = -\frac{LE_z}{R} = -\frac{LK\bar{u}B_y}{R}\Phi_{\text{open}} \tag{16}$$

Equating Eqs. (15) and (16) and substituting for Φ_{open} from Eq. (7) gives the results

$$R = \left(\frac{2R_i}{2+\theta_1+\theta_2}\right)\left(\frac{K}{1-K}\right) = R^*\left(\frac{K}{1-K}\right) \tag{17}$$

and

$$K = \frac{1}{1 + \{2R_i[R(2+\theta_1+\theta_2)]\}} = \frac{1}{1+(R^*/R)} \tag{18}$$

Equations (17) and (18) show the influence of the wall conductances on the relationship between external load and operating voltage ratio. Equating Eq. (18) to the values of K, which optimize the power output, then establishes the relationship between external load resistance and internal resistance of fluid and channel walls which must be satisfied for maximum power output. For constant mass flow rate, the ratio of internal to external resistance which maximizes the power output is

$$R^*/R = 1 \tag{19}$$

For constant pressure gradient operation, the corresponding ratio is

$$\frac{R^*}{R} = 1 + \frac{2(\tanh M - M)}{M(2+\theta_1+\theta_2)} \tag{20}$$

References

[1] Chang, C. C. and Lundgren, T. S., "The flow of an electrically conducting fluid through a duct with transverse magnetic field," *Proceedings Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, Calif., 1959), pp. 41–54.

[2] Chang, C. C. and Yen, J. T. "Magnetohydrodynamic channel flow as influenced by wall conductance," Z. Angew. Math. Phys. **XIII**, 266–272 (1962).

Transient Radiation Heating of a Rotating Cylindrical Shell

W. E. Olmstead,* L. A. Peralta,† and S. Raynor‡
Northwestern University, Evanston, Ill.

Nomenclature

r	= radius of the cylinder, ft
s	= wall thickness (assumed to be small compared with the radius), ft
k	= thermal conductivity of the wall material, Btu/ft hr °R
σ	= Stefan Boltzmann constant = 0.1717×10^{-8} Btu/ft² hr °R⁴
c	= specific heat of the wall material, Btu/lb °R
ρ	= density of the wall material, lb/ft³
a	= average (with respect to wave length and angle of incidence) absorptivity of the wall material
e	= average total hemispherical emissivity of the wall material for the spectrum of wave length radiated by the cylinder
K_s	= energy received from the sun by a plane perpendicular to the "line of vision," Btu/ft²hr
$K_N(t)$	= energy received from the thermal radiation pulse by a plane perpendicular to the "line of vision," Btu/ft²hr
ψ	= angular position fixed with respect to the rotation cylinder
θ	= angular position fixed with respect to the heat sources

Introduction

A BRIEF survey of the literature pertaining to the heating of rotating shells by radiation can be found in Refs. 1 and 2. The problem considered here is that of a thin-walled circular cylinder, rotating with uniform velocity about its geometric axis. Initially, it has a temperature distribution corresponding to the equilibrium state with the solar radiation. It is then suddenly exposed to a time-dependent source of radiation. It is assumed that no heat exchange takes place inside of the cylinder and that external heat losses occur by thermal radiation only. Furthermore, it is assumed that the time-dependent source is on a "line of vision" between the cylinder and the sun (the equations easily can be

Received March 25, 1963; revision received July 8, 1963.

* Research Fellow, Department of Mechanical Engineering and Astronautical Sciences.

† Member of Technical Staff, The Mitre Corporation.

‡ Professor, Department of Mechanical Engineering and Astronautical Sciences.

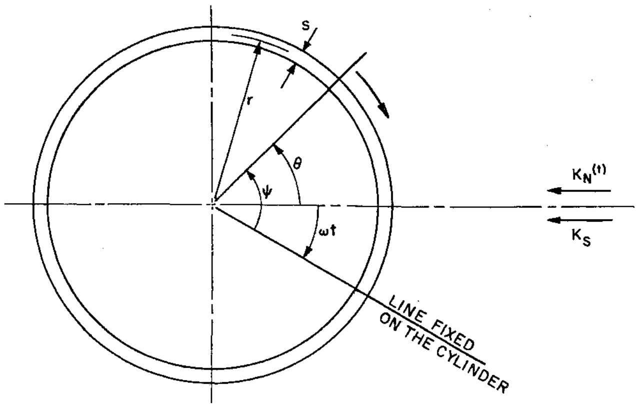

Fig. 1 The coordinate system.

modified for arbitrary position of the sun). The distances between these two heat sources and the axis of the cylinder are considered constant and very large compared with the cylinder diameter and cylinder length. The length of the cylinder is large compared with the cylinder diameter so that no heat flow takes place in the axial direction, and therefore all calculations are made per unit length of the cylinder.

Differential Equation and Solution

The governing differential equation for the temperature as defined in a frame of reference fixed relative to the heat sources (Fig. 1) is

$$\frac{\partial^2 T}{\partial\theta^2} - \frac{\sigma e r^2}{ks} T^4 + \frac{r^2 a K_N(t)}{ks}\cos^+\theta + \frac{r^2 a K_s}{ks}\cos^+\theta = \frac{\rho c r^2}{k}\left(-\omega\frac{\partial T}{\partial\theta} + \frac{\partial T}{\partial t}\right) \quad (1)$$

where

$$\cos^+\theta = \begin{cases}\cos\theta & -(\pi/2) \le \theta \le (\pi/2)\\ 0 & (\pi/2) \le \theta \le (3\pi/2)\end{cases}$$

Define

$$T(\theta,t) = T_s(\theta)[1 + \hat{T}(\theta,t)] \quad (2)$$

where $\hat{T}(\theta,t) \ll 1$, and $T_s(\theta)$ is the steady state temperature distribution due to solar radiation. Hence, the initial condition follows:

$$\hat{T}(\theta,0) = 0 \quad (3)$$

In Ref. 1, the steady state temperature distribution due to solar radiation is expressed as

$$T_s(\theta) = T_0[1 + \hat{T}_s(\theta)] \quad (4)$$

and the solution is obtained in closed form for $\hat{T}_s(\theta) \ll 1$, with the average temperature determined as

$$T_0 = (aK_s/\pi\sigma e)^{1/4} \quad (5)$$

Neglecting second-order terms, it follows from Eq. (2) that

$$T(\theta,t) \approx T_0[1 + \hat{T}_s(\theta) + \hat{T}(\theta,t)] \quad (6)$$

$$T^4(\theta,t) \approx 4T_0^4[\tfrac{1}{4} + \hat{T}_s(\theta) + \hat{T}(\theta,t)] \quad (7)$$

Substitution into the differential equation yields

$$\left\{T_0\frac{d^2\hat{T}_s}{d\theta^2} - \frac{4\sigma e r^2 T_0^4}{ks}\left(\frac{1}{4} + \hat{T}_s\right) + \frac{\rho c r^2\omega T_0}{k}\frac{d\hat{T}_s}{d\theta} + \frac{r^2 a K_s}{ks}\cos^+\theta\right\} + \left\{T_0\frac{\partial^2\hat{T}}{\partial\theta^2} - \frac{4\sigma e r^2 T_0^4}{ks}\hat{T} + \frac{\rho c r^2 \omega T_0}{k}\frac{\partial\hat{T}}{\partial\theta} - \frac{\rho c r^2 T_0}{k}\frac{\partial\hat{T}}{\partial t} + \frac{r^2 a K_N(t)}{ks}\cos^+\theta\right\} = 0 \quad (8)$$

The quantity in the first pair of braces is identically zero, since it constitutes the differential equation for the temperature variation due to solar radiation as derived in Ref. 1. Thus remains the differential equation for the temperature variation due to time-dependent radiation source,

$$\frac{\partial^2\hat{T}}{\partial\theta^2} + \frac{\omega}{\beta}\frac{\partial\hat{T}}{\partial\theta} - \frac{\alpha}{\beta}\hat{T} - \frac{1}{\beta}\frac{\partial\hat{T}}{\partial t} = -\frac{\pi\mu}{\beta}f(t)\cos^+\theta \quad (9)$$

where

$$\alpha = 4\sigma e T_0^3/\rho c s \qquad \beta = k/\rho c r^2$$
$$\mu = aN/\pi T_0\rho c s \qquad K_N(t) = Nf(t)$$

The quantity N is some dimensional constant (Btu/ft²hr), which is to be specified for a particular problem.

Performing a Laplace transform with respect to time on Eq. (9) with initial condition given by Eq. (3) yields

$$\frac{d^2\tau}{d\theta^2} + \frac{\omega}{\beta}\frac{d\tau}{d\theta} - \left(\frac{\alpha}{\beta} + \frac{p}{\beta}\right)\tau = -\frac{\pi\mu}{\beta}\bar{f}(p)\cos^+\theta \quad (10)$$

where $L\{\hat{T}(\theta,t)\} = \tau(\theta,p)$ and $L\{f(t)\} = \bar{f}(p)$.

Since T, $\hat{T}$, τ are periodic functions of θ, τ can be expanded in a complex Fourier series in θ as

$$\tau(\theta,p) = \sum_{n=-\infty}^{\infty} C_n(p)e^{in\theta} \quad (11)$$

$C_n(p)$ can be computed from Eq. (10) by multiplying by $e^{-in\theta}$ and integrating with respect to θ in the limits from 0 to 2π, yielding

$$C_n(p) = \frac{\frac{1}{2}\mu\bar{f}(p)}{n^2\beta + \alpha - in\omega + p}\int_0^{2\pi}\cos^+\theta' e^{-in\theta'}d\theta' \quad (12)$$

Substituting Eq. (12) into Eq. (11) and formally inverting the transform yields

$$\hat{T}(\theta,t) = \frac{\mu}{2}\sum_{n=-\infty}^{\infty}\left(\int_0^{2\pi}\cos^+\theta' e^{-in\theta'}d\theta'\right) \times L^{-1}\left[\frac{\bar{f}(p)}{n^2\beta + \alpha - in\omega + p}\right]e^{in\theta} \quad (13)$$

Carrying out the integration indicated in Eq. (13) and inverting the transform by means of the convolution theorem yields the following expression for the transient temperature distribution:

$$T(\theta,t) = T_s(\theta)\left\{1 + \mu a_0(t)e^{-\alpha t} + \frac{\mu\pi}{2}e^{-(\beta+\alpha)t} \times [a_1(t)\cos(\theta + \omega t) + b_1(t)\sin(\theta + \omega t)] + 2\mu\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2 - 1}e^{-(4n^2\beta+\alpha)t}[a_{2n}(t)\cos 2n(\theta + \omega t) + b_{2n}(t)\sin 2n(\theta + \omega t)]\right\} \quad (14)$$

where

$$a_n(t) = \int_0^t f(\lambda)e^{(\beta n^2+\alpha)\lambda}\cos n\omega\lambda\, d\lambda \quad (15)$$

$$b_n(t) = \int_0^t f(\lambda)e^{(\beta n^2+\alpha)\lambda}\sin n\omega\lambda\, d\lambda \quad (16)$$

Numerical Example for a Nuclear Weapon

As a particular example of the forementioned, consider the time-dependent radiation as that of a nuclear weapon. A nondimensional plot of the power output vs time is given in Ref. 3. It was found that a good curve fit resulted from a series of six terms of the form

$$f(t) = \sum_{j=1}^{6} G_j e^{-(3j-2)\delta_1\cdot t/t_M} \quad (17)$$

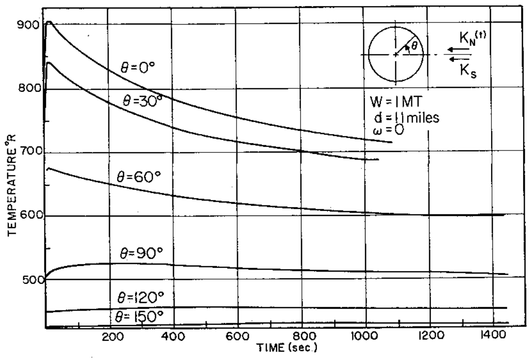

Fig. 2 Temperature distribution in a nonrotating cylinder.

with the constant N given as

$$N = P_{max}/4\pi d^2$$

P_{max} is the maximum radiative power of the weapon, and t_M is the time required to achieve that value. The distance from the cylindrical shell to the weapon is given by d.

For the curve fit, the value $\delta_1 = 0.35$ was chosen to give proper behavior as $t \to \infty$. The values of G_j are $G_1 = 0.574$, $G_2 = 0.090$, $G_3 = 39.181$, $G_4 = -146.052$, $G_5 = 176.776$, and $G_6 = -70.568$.

Using Eq. (17), the coefficients $a_n(t)$ and $b_n(t)$ can be evaluated by quadrature of Eqs. (15) and (16):

$$a_n(t) = \sum_{j=1}^{6} \frac{G_j}{(\beta n^2 + \alpha - \gamma_j)^2 + n^2\omega^2} \times \{e^{(\beta n^2+\alpha-\gamma_j)t} [(\beta n^2 + \alpha - \gamma_j) \cos n\omega t + n\omega \sin n\omega t] - (\beta n^2 + \alpha - \gamma_j)\} \quad (18)$$

$$b_n(t) = \sum_{j=1}^{6} \frac{G_j}{(\beta n^2 + \alpha - \gamma_j)^2 + n^2\omega^2} \times \{e^{(\beta n^2+\alpha-\gamma_j)t} [(\beta n^2 + \alpha - \gamma_j) \sin n\omega t - n\omega \cos n\omega t] + n\omega\} \quad (19)$$

where

$$\gamma_j = \frac{\delta_1(3j - 2)}{t_M} \qquad \beta n^2 + \alpha - \gamma_j \neq 0$$

The parameters describing the cylindrical shell were chosen as $r = 1$ ft, $s = \frac{1}{200}$ ft, $k = 100$ Btu/ft hr°R, $k\rho/c = 3$ ft²/hr, and $K_s = 442$ Btu/ft²hr. From this follows: $T_0 = 535.1$ °R, $\alpha = 0.00175$ sec^{-1}, and $\beta = 0.000833$ sec^{-1}.

Specifying a 1-megaton weapon, the formulas of Ref. 3 give $P_{max} = 126.5 \times 10^{12}$ cal/sec and $t_M = 1$ sec. Further, choosing $d = 11$ miles gives $\mu = 0.0422$ sec^{-1}. With these numerical values the temperature distribution was computed for three special cases.

For the case of zero speed ($\omega = 0$), the temperature distribution was computed and plotted in Fig. 2 for several values of angle θ. At time zero, the distribution is that given by Ref. 1 for solar heating at zero speed.

For the case of very high speed ($\omega \to \infty$), the initial temperature is uniform, i.e., $T_s = T_0 = 535.1$ °R. The temperature variation with time is identical for all positions ψ on the cylinder. This variation is plotted in Fig. 3.

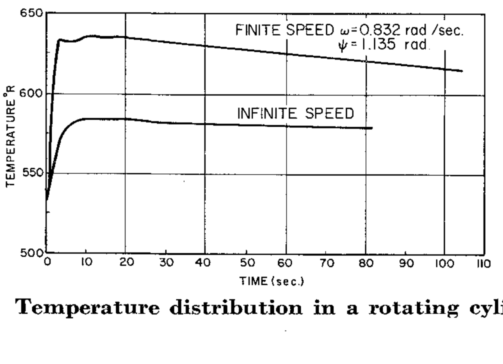

Fig. 3 Temperature distribution in a rotating cylinder.

For the case of finite speed ($\omega = 0.832$ rad/sec), the initial temperature distribution is still essentially uniform. The point on the cylinder which attains the highest temperature is located initially by $\psi = 1.135$ rad. The temperature history of this position is given in Fig. 3.

References

[1] Charnes, A. and Raynor, S., "Solar heating of a rotating cylindrical space vehicle," ARS J. **30**, 479–484 (1960).

[2] Olmstead, W. E. and Raynor, S., "Solar heating of a rotating spherical space vehicle," Intern. J. Heat Mass Transfer **5**, 1165–1177 (1962).

[3] Glasstone, S., "The effects of nuclear weapons," U. S. Dept. Defense, U. S. Atomic Energy Commission, U. S. Government Printing Office: 1957 0-424278 (June 1957).

Sphere of Influence in Patched-Conic Methods

Jack W. Crenshaw*
General Electric Company, Daytona Beach, Fla.

The so-called "patched-conic" method as an approximation of space trajectories is well known and widely used. In this method, each portion of the trajectory is treated as a two-body trajectory about the most influential body. At some point on a trajectory between two attracting bodies, these two bodies are equally influential, and the locus of such points is a roughly spheroidal surface. This surface is called the "sphere of influence," although, as will be seen, it is not truly spherical.

At this sphere of influence, a transition is made from one attracting body to the other by matching the relative position and velocity in the appropriate conic trajectories. Obviously, the choice of the surface at which the matching takes place is important to the overall result. A philosophy for choosing this surface can be developed by considering the desired end result: that the errors be minimized. This criterion can be stated mathematically, and other authors[1–3] have derived the approximate surface that results. (For the most complete such derivation, see Plummer.[9]) For practical reasons, the resulting spheroid almost always is further approximated by a sphere.[4–8,10]

If one is to use the patched-conic method with confidence, it is imperative that one be able to place bounds on the errors incurred. This means that one must examine each simplifying assumption and its effect. One such assumption is that of the shape of the sphere of influence, and for this reason an exact determination of its shape is desirable.

In studying the patched conic method, one need consider only three bodies at a time, since the sphere of influence always is determined by the two major attracting bodies. Figure 1 shows a three-body system (M_1, M_2, V) that is restricted only in the sense that the mass of the vehicle V is assumed negligible. It is assumed that the mass M_1 is greater than M_2. The radius vector convention uses capital letters to denote vectors from M_1 and lower case letters to denote those from M_2. The angle θ is defined by the dot product

$$\bar{r} \cdot \bar{R}_2 = -rR_2 \cos\theta \qquad 0 \leq \theta \leq \pi \quad (1)$$

Received April 15, 1963.

* Physicist, Flight Dynamics, Apollo Support Department. Member AIAA.